

ALMOST FINITENESS FOR GENERAL ÉTALE GROUPOIDS AND ITS APPLICATIONS TO STABLE RANK OF CROSSED PRODUCTS

YUHEI SUZUKI

ABSTRACT. We extend Matui's notion of almost finiteness to general étale groupoids. We then show that the reduced groupoid C^* -algebras of minimal almost finite groupoids have stable rank one. The proof follows a new strategy, which can be regarded as a local version of the large subalgebra argument.

We present two applications to the reduced crossed products. Firstly, for any minimal action of any abelian group on a compact space with a totally disconnected free factor, we show that the crossed product has stable rank one. In the finitely generated case, this is the main results of recent works of Archey–Phillips and Phillips. Secondly, we show that any countable amenable group admits a minimal action on the Cantor set whose minimal extension always has the crossed product of stable rank one.

1. INTRODUCTION

In the seminal paper [23], Rieffel introduced the notion of stable rank for C^* -algebras. The stable rank is introduced as a C^* -algebraic analog of the Lebesgue covering dimension. This in fact detects a topological complexity of C^* -algebras in the sense that when a given C^* -algebra has stable rank one (the smallest possible value), its K -theory behaves well.

In the celebrated paper [28], Villadsen showed that simple stably finite C^* -algebras can have arbitrary stable rank. In fact, Villadsen has constructed such C^* -algebras as AH-algebras. His work suggests that even for relatively tractable C^* -algebras, it would be hard to compute their stable rank.

In the present paper, we give a new strategy to compute the stable rank of certain groupoid C^* -algebras. This strategy may be regarded as a local version of the large subalgebra argument ([21], [22], [18], [19], [20]). To describe the difference between the large subalgebra argument and our strategy, let us briefly recall how the large subalgebra argument works. For more detailed explanations and applications of large subalgebras, we refer the reader to the introduction of the paper [19]. The strategy can be divided into three steps. Firstly, we find a large C^* -subalgebra. Secondly, we study the structure of the large subalgebra. Finally, we analyze the reduced crossed product based on the large subalgebra. For the last step, Phillips [19] and Archey–Phillips [3] recently developed general theories. For the first step, when the group is \mathbb{Z} , large subalgebras of the crossed products can be found as Putnam's orbit-breaking subalgebras [21], [19]. For the group \mathbb{Z}^d ; $d \in \mathbb{N}$, Phillips [18] has found that Forrest's result [9] is sometimes useful to find a large subalgebra of the crossed product. However, Forrest's result [9] heavily uses the fact that \mathbb{Z}^d is a lattice of the Euclidean space \mathbb{R}^d . Therefore, for groups beyond finitely generated abelian groups (even for \mathbb{Z}^∞), the first step of the strategy would be very hard to carry out. Our strategy allows us to avoid this difficulty; instead of finding a single large subalgebra, we find a family of homogeneous C^* -algebras tending to large in a sense.

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Inspired by the work of Latrémolière–Orme [14], Matui [15] has introduced the notion of almost finiteness for totally disconnected étale groupoids in the study of homology groups and topological full groups. For further applications of almost finiteness in this direction, we refer the reader to [16] and [17] for instance. We extend the notion of almost finiteness to general étale groupoids, and adapt this notion in our strategy. Consequently, we obtain the following Main Theorem of the present paper.

Main Theorem. *Let G be a minimal almost finite étale groupoid. Then the reduced groupoid C^* -algebra $C_r^*(G)$ has stable rank one.*

We close this section by presenting applications of the Main Theorem.

Corollary I. *Let $\alpha: \Gamma \curvearrowright X$ be a minimal action of a discrete abelian group Γ on a compact space which induces a faithful action on the lattice of clopen subsets of X . Then the reduced crossed product $C(X) \rtimes_{r,\alpha} \Gamma$ has stable rank one.*

When $\Gamma = \mathbb{Z}$, this is the main result of the recent work of Archey–Phillips [3]. Phillips [20] generalizes it to the case $\Gamma = \mathbb{Z}^d$, where d is an arbitrary natural number. We believe that even for these known cases, our new proof is conceptually and practically simpler. We also remark that, even in the simplest case $\Gamma = \mathbb{Z}$, Giol–Kerr [10] has constructed an action $\Gamma \curvearrowright X$ as in the statement whose crossed product does not absorb the Jiang–Su algebra \mathcal{Z} tensorially. In particular, Rørdam’s theorem [25] is not applicable to these crossed products.

We also have the following result for general countable amenable groups.

Corollary II. *Any countable infinite amenable group Γ admits a minimal free action on the Cantor set whose all minimal extensions have the crossed product of stable rank one.*

We remark that any minimal amenable (topologically) free action of a countable infinite group on a compact metrizable space admits surprisingly many minimal extensions [11], [27]. Minimal actions appeared in the statement are nothing but the ones recently constructed by Kerr [13]. Nevertheless, to the best knowledge of the author, Corollary II is a new phenomenon in the study of topological dynamical systems of amenable groups.

Organization of the paper. Section 2 is the preliminary section. We also fix the notations in this section. In Section 3, we introduce and study the notion of almost finiteness for general étale groupoids. In Section 4, we prove the Main Theorem. In Section 5, we discuss applications of the Main Theorem to the crossed products of topological dynamical systems. Finally, in Appendix A, based on a technique developed in Section 4, we study the pure infiniteness of the reduced groupoid C^* -algebras. As a consequence, we solve the question of the pure infiniteness of the reduced crossed product for minimal extensions.

2. PRELIMINARIES

In this section, we recall some terminologies used in the paper. We first fix the notations used in the paper.

Notations.

- For a groupoid G , its range and source maps are denoted by r and s respectively.
- The unit space of a groupoid G is denoted by $G^{(0)}$.
- For two subsets U, V of a groupoid G , define

$$UV := \{uv : u \in U, v \in V, s(u) = r(v)\}.$$

When V is a singleton $\{v\}$, we simply denote UV by Uv .

- For a subset U of a groupoid G , define

$$U^{-1} := \{u^{-1} : u \in U\}.$$

- For a set S , $\sharp(S)$ denotes the cardinality of S .
- For a subset S of a set X , χ_S denotes the characteristic function of S .
- For a subset S of a topological space X , the symbols $\text{cl}(S)$ and $\text{int}(S)$ stand for the closure and interior of S respectively.
- For a map $\tau: A \rightarrow B$ between subsets of a set X and for a subset $U \subset X$, we define $\tau(U) := \tau(U \cap A)$. Similarly, for such a map τ and a function $f: X \rightarrow \mathbb{C}$, we define the function $f \circ \tau: X \rightarrow \mathbb{C}$ to be

$$(f \circ \tau)(s) := \begin{cases} (f \circ \tau)(s) & \text{if } s \in A \\ 0 & \text{otherwise.} \end{cases}$$

- For a group action $\alpha: \Gamma \curvearrowright X$ on a set X , we denote $\alpha_s(x)$ by sx for short. Similarly, for subsets $S \subset \Gamma$ and $U \subset X$, we denote $\bigcup_{s \in S} \alpha_s(U)$ by SU .
- For a probability space (X, μ) and a μ -integrable function f , we denote the integral $\int_X f d\mu$ by $\mu(f)$ for short.
- For a compact space X and an open subset $U \subset X$, we identify $C_0(U)$ with the ideal of $C(X)$ consisting of all functions vanishing on $X \setminus U$ in the canonical way.
- For a C^* -algebra A and $n \in \mathbb{N}$, $M_n(A)$ denotes the C^* -algebra of all n -by- n matrices over A .
- For a unital C^* -algebra A , denote $GL(A)$ the group of invertible elements in A . The symbol $\overline{GL}(A)$ stands for the norm closure of $GL(A)$ in A .

2.1. Groupoids. Throughout the paper, groupoids are supposed to be locally compact, Hausdorff, étale, and their unit spaces are supposed to be compact (possibly non-metrizable). We refer the reader to Section 5.6 of [4] for the definition and basic facts of groupoids.

Let G be a groupoid. Let $C_c(G)$ denote the space of compactly supported complex valued continuous functions on G . We equip $C_c(G)$ with the convolution product. Since we do not consider the pointwise product, we simply denote the convolution product of two functions $f, g \in C_c(G)$ by fg . Namely, for $f, g \in C_c(G)$ and $x \in G$,

$$(fg)(x) = \sum_{yz=x} f(y)g(z).$$

The involution on $C_c(G)$ is given by $f^*(x) = \overline{f(x^{-1})}$ for $f \in C_c(G)$ and $x \in G$. These operations make $C_c(G)$ a $*$ -algebra. The reduced groupoid C^* -algebra $C_r^*(G)$ of G is the unique C^* -completion of the $*$ -algebra $C_c(G)$ which admits a (unique) faithful conditional expectation

$$E: C_r^*(G) \rightarrow C(G^{(0)})$$

satisfying $E(f) = f|_{G^{(0)}}$ for all $f \in C_c(G)$. The E is referred to as the canonical conditional expectation of $C_r^*(G)$. We regard $C_c(G)$ as a $*$ -subalgebra of $C_r^*(G)$. With this identification, positivity and norm make a sense for elements of $C_c(G)$.

A subset U of $G^{(0)}$ is said to be G -invariant if we have $r(GU) = U$. A groupoid G is said to be minimal if there is no proper open G -invariant subset of $G^{(0)}$. A groupoid G is said to be principal (resp. essentially principal) if the set (resp. the interior of the set)

$$\{g \in G \setminus G^{(0)} : r(g) = s(g)\}$$

is empty. These notions have a strong connection with the simplicity of the reduced groupoid C^* -algebras as follows. When the reduced groupoid C^* -algebra is simple, then

obviously the groupoid must be minimal. Conversely, when a groupoid is minimal and essentially principal, the reduced groupoid C*-algebra is known to be simple [2].

A subset V of G is said to be a G -set if both the range and source maps are injective on V . For any compact G -set V , the formula $\tau_V := r \circ (s|_V)^{-1}$ defines a homeomorphism from $s(V)$ onto $r(V)$. Set

$$[[G]] := \{\tau_V : V \text{ is a compact open } G\text{-set with } r(V) = s(V) = G^{(0)}\}.$$

Then $[[G]]$ is a subgroup of the homeomorphism group $\text{Homeo}(G^{(0)})$.

We say that two clopen subsets U, V of $G^{(0)}$ are equivalent in G if there is a compact open G -set W satisfying $r(W) = U$ and $s(W) = V$. We denote this equivalence relation by \sim . We remark that when these U, V are disjoint, one can find an element $\varphi \in [[G]]$ satisfying $\varphi(U) = V$ and $\varphi(V) = U$. Indeed, the G -set $W \sqcup W^{-1} \sqcup (G^{(0)} \setminus U \sqcup V)$ defines such a φ . Suppose G is essentially principal and let $\varphi \in [[G]]$ be given. Take a compact open G -set V with $\varphi = \tau_V$, which is unique since G is essentially principal. We then define $u_\varphi := \chi_V$. Clearly u_φ is a unitary element of $C_r^*(G)$.

A probability regular Borel measure μ on $G^{(0)}$ is said to be G -invariant if for any compact G -set V and any Borel subset B of $s(V)$, we have $\mu(\tau_V(B)) = \mu(B)$. Note that by the uniqueness statement of the Riesz representation theorem, one can show that a probability regular Borel measure μ on $G^{(0)}$ is G -invariant if and only if for any compact G -set V and any continuous function $f \in C(G^{(0)})$ supported on $r(V)$, we have $\mu(f) = \mu(f \circ \tau_V)$. Let $M(G)$ denote the set of all G -invariant probability measures. Obviously $M(G)$ is compact with respect to the weak-* topology. Every $\mu \in M(G)$ defines a tracial state on the reduced groupoid C*-algebra by the formula $\mu \circ E$. Since E is faithful, the resulting tracial state is faithful if and only if the support of μ is $G^{(0)}$.

For a nonempty clopen subset U of $G^{(0)}$, set

$$G_U := r^{-1}(U) \cap s^{-1}(U).$$

Then the relative topology makes G_U an étale groupoid.

A subset U of $G^{(0)}$ is said to be G -full if it satisfies the equality $r(GU) = G^{(0)}$. Note that for any open G -full subset U , by the compactness of $G^{(0)}$, one can find a finite sequence V_1, \dots, V_n of compact G -sets satisfying $\bigcup_{i=1}^n \tau_{V_i}(U) = G^{(0)}$.

2.2. Topological dynamical systems. Motivating examples of groupoids in the present paper are coming from topological dynamical systems. From now on, the symbol Γ stands for a discrete group (possibly uncountable), and the symbols X, Y stand for compact Hausdorff spaces (possibly nonmetrizable). Actions of Γ on X, Y are always supposed to be continuous. For a given action $\alpha: \Gamma \curvearrowright X$, one can associate a groupoid $X \rtimes_\alpha \Gamma$, the transformation groupoid of α . The transformation groupoid $X \rtimes_\alpha \Gamma$ is obtained by equipping the product space $\Gamma \times X$ with the following groupoid structure. First, the range and source maps r and s are defined to be $r(g, x) := \alpha_g(x)$ and $s(g, x) := x$ respectively. Then, for a composable pair $((g, x), (h, y))$ (i.e., in the case $x = \alpha_h(y)$), the composite $(g, x)(h, y)$ is defined to be (gh, y) . The reduced groupoid C*-algebra of $X \rtimes_\alpha \Gamma$ is naturally isomorphic to the reduced crossed product $C(X) \rtimes_{r, \alpha} \Gamma$ of α .

An action $\alpha: \Gamma \curvearrowright X$ is said to be minimal if the transformation groupoid is minimal. An action α is said to be free (resp. essentially free) if the transformation groupoid is principal (resp. essentially principal). For two actions $\alpha: \Gamma \curvearrowright X$ and $\beta: \Gamma \curvearrowright Y$, α is said to be an extension of β if there is a Γ -equivariant quotient map $\pi: X \rightarrow Y$. In this case β is said to be a factor of α . Note that any Γ -equivariant quotient map $\pi: X \rightarrow Y$ extends to a proper quotient homomorphism $\pi: X \rtimes_\alpha \Gamma \rightarrow Y \rtimes_\beta \Gamma$ by the formula $\pi(s, x) := (s, \pi(x))$.

2.3. Stable rank. Here we recall the definition and basic facts of stable rank for unital C^* -algebras. We refer the reader to [23] for further information.

Definition 2.1. Let A be a unital C^* -algebra. For each $n \in \mathbb{N}$, define

$$\text{Lg}_n(A) := \{(a_1, \dots, a_n) \in A^n : Aa_1 + \dots + Aa_n = A\}.$$

Then the stable rank of A is defined to be the value

$$\min\{n \in \mathbb{N} : \text{Lg}_n(A) \text{ is norm dense in } A^n\}.$$

Obviously, the stable rank takes the value in $\{1, 2, \dots, \infty\}$. For any $n \in \{1, 2, \dots, \infty\}$, Villadsen [28] has constructed a simple unital separable AH-algebra of stable rank n .

For the stable rank one, Rieffel has found the following useful characterization.

Proposition 2.2 ([23], Proposition 3.1). *A unital C^* -algebra A has stable rank one if and only if the set of invertible elements of A is norm dense in A .*

Rieffel gave the following striking application of the stable rank one in the K-theory.

Theorem 2.3 ([23]). *Let A be a unital C^* -algebra of stable rank one. Let $n \in \mathbb{N}$ be given. Then the following statements hold true.*

- (1) *For two projections p, q of $M_n(A)$, they are unitary equivalent in $M_n(A)$ if and only if we have $[p]_0 = [q]_0$ in $K_0(A)$.*
- (2) *For two unitary elements u, v of $M_n(A)$, they sit in the same connected component of the unitary group of $M_n(A)$ if and only if we have $[u] = [v]$ in $K_1(A)$.*

This emphasizes the importance of the question when a given C^* -algebra has stable rank one. In fact, using Theorem 2.3 (and the Main Theorem), one can show that certain groupoid C^* -algebras have real rank zero and strict comparison (see Remark 4.3).

2.4. Connected components and separations by clopen subsets. Let X be a compact space. For $x \in X$, define $\langle x \rangle$ to be the intersection of all clopen neighborhoods of x . Observe that any clopen subset of X does not separate elements in $\langle x \rangle$. The following lemmas are immediate consequences of the definition.

Lemma 2.4. *Let $x \in X$ be given. Let C be a closed subset of X disjoint to $\langle x \rangle$. Then there is a clopen subset $U \subset X$ satisfying $\langle x \rangle \subset U \subset X \setminus C$.*

Lemma 2.5. *Let x_1, \dots, x_n be elements of X satisfying $\langle x_i \rangle \neq \langle x_j \rangle$ for $i \neq j$. Then there are pairwise disjoint clopen subsets U_1, \dots, U_n of X satisfying $x_i \in U_i$ for each i .*

3. ALMOST FINITE GROUPOIDS

In this section, we introduce a notion of almost finiteness for general groupoids. This notion is originally introduced for totally disconnected groupoids by Matui [15]. We then establish basic properties of almost finite groupoids which are used in Section 4.

We first introduce a few terminologies needed to define almost finiteness.

Definition 3.1. Let C be a compact subset of G and let $\epsilon > 0$. Let K be a compact subgroupoid of G containing the unit space $G^{(0)}$. We say that K is (C, ϵ) -invariant if the following inequality holds for all $s \in G^{(0)}$.

$$\frac{\#(CKs \setminus Ks)}{\#(Ks)} < \epsilon.$$

Definition 3.2. Let K be a compact groupoid. We say that a clopen subset F of $K^{(0)}$ is a fundamental domain of K if there is a partition $K^{(0)} = \bigsqcup_{i=1}^n \bigsqcup_{j=1}^{N_i} F_j^{(i)}$ of $K^{(0)}$ by clopen subsets satisfying the following conditions.

- (1) $F = \bigsqcup_{i=1}^n F_1^{(i)}$.
- (2) For each $i \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, N_i\}$, there is a unique compact open K -set $V_{j,k}^{(i)}$ satisfying $r(V_{j,k}^{(i)}) = F_j^{(i)}$ and $s(V_{j,k}^{(i)}) = F_k^{(i)}$.
- (3) $K = \bigsqcup_{i=1}^n \bigsqcup_{1 \leq j, k \leq N_i} V_{j,k}^{(i)}$.

Note that K is isomorphic to the groupoid $\coprod_{i=1}^n F_1^{(i)} \times (\mathbb{Z}_{N_i} \ltimes \mathbb{Z}_{N_i})$. Here for each i , the group \mathbb{Z}_{N_i} acts on itself by the left translations. From this description, it is easy to show that the analog of Lemma 6.1 in [15] is valid for compact groupoids admitting a fundamental domain.

Definition 3.3. Let G be an étale groupoid. We say that a subgroupoid K of G is elementary if K contains the unit space $G^{(0)}$ and admits a fundamental domain.

We now introduce the definition of almost finiteness for general étale groupoids.

Definition 3.4 (Cf. Definition 6.2 of [15]). We say that a groupoid G is almost finite if it satisfies the following conditions.

- (1) The union of all compact open G -sets covers G .
- (2) For any compact subset $C \subset G$ and $\epsilon > 0$, there is a (C, ϵ) -invariant elementary subgroupoid K of G .

We remark that, by Lemma 4.7 of [15], for totally disconnected groupoids, our definition of almost finiteness coincides with Matui's original one.

We first study a connection between invariant probability measures of almost finite groupoids and that of almost invariant elementary subgroupoids.

By modifying the proof of Lemma 6.5 in [15] in a straightforward way, one can show the following lemma. Here we omit the proof. See [15] for the details.

Lemma 3.5 (Cf. Lemma 6.5 in [15]). *Let G be a groupoid. Let V be a compact G -set and let $\epsilon > 0$. Let K be a $(V \cup V^{-1}, \epsilon)$ -invariant elementary subgroupoid of G . Then for any $\mu \in M(K)$, we have*

$$|\mu(r(V \setminus K))| < \epsilon \text{ and } |\mu(s(V \setminus K))| < \epsilon.$$

In particular, for any Borel subset A of $s(V)$, we have

$$|\mu(A) - r(\tau_V(A))| < 2\epsilon.$$

Lemma 3.6 (Cf. Lemma 6.5 of [15]). *Let G be an almost finite groupoid. Let $(C_\lambda)_{\lambda \in \Lambda}$ be an increasing net of compact subsets of G satisfying $\bigcup_{\lambda \in \Lambda} \text{int}(C_\lambda) = G$. Let $(\epsilon_\lambda)_{\lambda \in \Lambda}$ be a net of positive numbers decreasing to 0. For each $\lambda \in \Lambda$, let a $(C_\lambda, \epsilon_\lambda)$ -invariant elementary subgroupoid K_λ of G and $\mu_\lambda \in M(K_\lambda)$ be given. Then any weak-* cluster point of the net $(\mu_\lambda)_{\lambda \in \Lambda}$ is G -invariant. In particular $M(G)$ is nonempty.*

Proof. Let μ be a weak-* cluster point of the net $(\mu_\lambda)_{\lambda \in \Lambda}$. It suffices to show the equality $\mu(f) = \mu(f \circ \tau_V)$ for any $f \in C(X)$ and any compact G -set V satisfying $\text{supp}(f) \subset r(V)$.

Given such f and V . Let $\epsilon > 0$ be given. Take a partition $\text{supp}(f) = \bigsqcup_{i=1}^n A_i$ of $\text{supp}(f)$ by Borel sets and a sequence $z_1, \dots, z_n \in \mathbb{C}$ satisfying $|f(s) - z_i| < \epsilon$ for all $s \in A_i$ and i . For each i , put $B_i := \tau_V^{-1}(A_i)$. Then, by Lemma 3.5, one can find $\lambda \in \Lambda$ satisfying the following conditions.

- (1) $|\mu(f) - \mu_\lambda(f)| < \epsilon.$
- (2) $|\mu(f \circ \tau_V) - \mu_\lambda(f \circ \tau_V)| < \epsilon.$
- (3) $\sum_{i=1}^n |z_i| |\mu_\lambda(A_i) - \mu_\lambda(B_i)| < \epsilon.$

By the choices of the partition $(A_i)_{i=1}^n$ and the sequence $(z_i)_{i=1}^n$, we have

$$\left| \mu_\lambda(f) - \sum_{i=1}^n z_i \mu_\lambda(A_i) \right| < \epsilon.$$

Since $\text{supp}(f \circ \tau_V) = \tau_V^{-1}(\text{supp}(f))$, we have

$$\text{supp}(f \circ \tau_V) = \bigsqcup_{i=1}^n B_i.$$

Moreover, for any $i \in \{1, \dots, n\}$ and any $s \in B_i$, we have $|(f \circ \tau_V)(s) - z_i| < \epsilon$. This yields

$$\left| \mu_\lambda(f \circ \tau_V) - \sum_{i=1}^n z_i \mu_\lambda(B_i) \right| < \epsilon.$$

By combining these inequalities, we obtain

$$\begin{aligned} |\mu(f) - \mu(f \circ \tau_V)| &\leq |\mu(f) - \mu_\lambda(f)| + |\mu_\lambda(f) - \mu_\lambda(f \circ \tau_V)| + |\mu(f \circ \tau_V) - \mu_\lambda(f \circ \tau_V)| \\ &< 4\epsilon + \sum_{i=1}^n |z_i| |\mu_\lambda(A_i) - \mu_\lambda(B_i)| \\ &< 5\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\mu(f) = \mu(f \circ \tau_V)$ as desired. \square

Lemma 3.7 (Cf. Remark 6.6 in [15]). *Any minimal almost finite groupoid is essentially principal.*

Proof. To lead to a contradiction, suppose we have a minimal almost finite groupoid G which is not essentially principal. Take a compact G -set $V \subset G \setminus G^{(0)}$ satisfying $\text{int}(V) \neq \emptyset$ and $r(g) = s(g)$ for all $g \in V$. Take $\mu \in M(G)$. For any $\epsilon > 0$, take a $(V \cup V^{-1}, \epsilon)$ -invariant elementary subgroupoid K of G . Then, since K is principal, we have $K \cap V = \emptyset$. This equality and Lemma 3.5 yield $\mu(s(V)) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\mu(s(V)) = 0$. This contradicts the minimality of G . \square

Lemma 3.8. *Let G be an almost finite groupoid. Let U be an open G -full set. Then there is an elementary subgroupoid K of G which makes U a K -full set.*

Proof. To lead to a contradiction, assume that such a K does not exist for a G -full open set U . We observe that for any elementary subgroupoid K of G , U is K -full if and only if it satisfies $\mu(U) > 0$ for all $\mu \in M(K)$. Take nets $(C_\lambda)_{\lambda \in \Lambda}$, $(\epsilon_\lambda)_{\lambda \in \Lambda}$, and $(K_\lambda)_{\lambda \in \Lambda}$ as in the statement of Lemma 3.6. Then by assumption, for each $\lambda \in \Lambda$, one can find $\mu_\lambda \in M(K_\lambda)$ satisfying $\mu_\lambda(U) = 0$. Let μ be a weak-* cluster point of the net $(\mu_\lambda)_{\lambda \in \Lambda}$. By Lemma 3.6, we have $\mu \in M(G)$. Since each μ_λ is supported on $G^{(0)} \setminus U$, so μ is. This contradicts the G -fullness of U . \square

Lemma 3.9. *Let G be an almost finite groupoid. Let U be a G -full clopen subset of $G^{(0)}$. Then the restriction groupoid G_U is almost finite.*

Proof. Since U is G -full, one can take compact G -sets V_1, \dots, V_n satisfying $r(V_i) \subset U$ for all i and $\bigcup_{i=1}^n s(V_i) = G^{(0)}$. Let a compact subset $C \subset G_U$ and $\epsilon > 0$ be given. We

assume that $\epsilon \leq 1$. Set $\tilde{C} := C \cup V_1 \cup V_2 \cup \cdots \cup V_n$. We claim that for any $(\tilde{C}, \epsilon/2n)$ -invariant elementary subgroupoid K of G , the restriction groupoid K_U is a (C, ϵ) -invariant elementary subgroupoid of G_U . This completes the proof.

Let K be a $(\tilde{C}, \epsilon/2n)$ -invariant elementary subgroupoid of G . Let $s \in U$ be given. To prove the claim, we first show the inequality

$$\sharp(K_U s) \geq \frac{1}{2n} \sharp(Ks).$$

The equality $\bigcup_{i=1}^n s(V_i) = G^{(0)}$ yields

$$\sum_{i=1}^n \sharp(V_i K s) \geq \sharp(Ks).$$

Hence one can choose $i \in \{1, \dots, n\}$ satisfying

$$\sharp(V_i K s) \geq \frac{1}{n} \sharp(Ks).$$

By the choice of K and the assumption $\epsilon \leq 1$, we have

$$\sharp(V_i K s \setminus Ks) \leq \frac{1}{2n} \sharp(Ks).$$

Since $V_i K s \subset G_U$, these two inequalities yield

$$\begin{aligned} \sharp(K_U s) &\geq \sharp(V_i K s \cap Ks) \\ &\geq \sharp(V_i K s) - \frac{1}{2n} \sharp(Ks) \\ &\geq \frac{1}{2n} \sharp(Ks). \end{aligned}$$

Since $CK_U s \setminus K_U s \subset \tilde{C}Ks \setminus Ks$, we conclude

$$\begin{aligned} \frac{\sharp(CK_U s \setminus K_U s)}{\sharp(K_U s)} &\leq 2n \frac{\sharp(\tilde{C}Ks \setminus Ks)}{\sharp(Ks)} \\ &< \epsilon. \end{aligned}$$

□

Lemma 3.9 in particular implies the following divisibility property for clopen subsets of minimal almost finite groupoids. This property plays an important role in the proof of the Main Theorem.

Lemma 3.10. *Let G be a minimal almost finite groupoid of infinite cardinality. Let $U \subset G^{(0)}$ be a clopen subset. Then, for any $N \in \mathbb{N}$, there is a partition $U = \bigsqcup_{i=1}^n \bigsqcup_{j=1}^{N_i} U_j^{(i)}$ of U by clopen subsets such that $U_j^{(i)} \sim U_k^{(i)}$ for all i, j, k , and that $N_i \geq N$ for all i .*

Proof. By Lemma 3.9, we only need to show the statement in the case $U = G^{(0)}$. In this case, the statement is equivalent to the existence of an elementary subgroupoid K of G satisfying $\sharp(Ks) \geq N$ for all $s \in G^{(0)}$.

To find such a subgroupoid, take a compact subset C of G satisfying $\sharp(Cs) \geq 2N$ for all $s \in G^{(0)}$. (To find such a C , take an increasing net $(C_\lambda)_{\lambda \in \Lambda}$ of compact open subsets of G whose union covers G . Then, for each $\lambda \in \Lambda$, the set

$$U_\lambda := \{s \in G^{(0)} : \sharp(C_\lambda s) \geq 2N\}$$

is clopen, and the union of the increasing net $(U_\lambda)_{\lambda \in \Lambda}$ covers $G^{(0)}$. By the compactness of $G^{(0)}$, for sufficiently large $\lambda \in \Lambda$, the compact set C_λ satisfies the desired condition.) Take a $(C, 1)$ -invariant elementary subgroupoid K of G . Then, for any $s \in G^{(0)}$, we have

$$\sharp(Cs) - \sharp(Ks) \leq \sharp(CKs \setminus Ks) \leq \sharp(Ks).$$

This yields $\sharp(Ks) \geq N$ for all $s \in G^{(0)}$. \square

Lemma 3.11 (Cf. [15], Lemma 6.7). *Let G be an almost finite groupoid. Let U and V be clopen subsets of $G^{(0)}$ satisfying $\mu(U) < \mu(V)$ for all $\mu \in M(G)$. Then there is an element $\varphi \in [[G]]$ with $\varphi(U) \subset V$.*

Proof. As mentioned in Definition 3.2, Lemma 6.1 of [15] is valid for any groupoid admitting a fundamental domain. This fact and Lemma 3.6 allow us to prove the claim in the same way as Lemma 6.7 of [15]. See [15] for the details. \square

4. STABLE RANK OF C^* -ALGEBRAS OF ALMOST FINITE GROUPOIDS

In this section, we prove the Main Theorem. Since the Main Theorem is obviously valid for groupoids of finite cardinality, throughout this section, we suppose that groupoids are of infinite cardinality.

We first prove the following key lemma, which is about zero divisors of the reduced groupoid C^* -algebras. Recall that an element a of a ring R is called a right (resp. left) zero divisor if there is a nonzero element $x \in R$ satisfying $xa = 0$ (resp. $ax = 0$). An element of R is called a two-sided zero divisor if it is both a right and a left zero divisor.

Lemma 4.1. *Let G be a minimal almost finite groupoid. Let $a \in C_r^*(G)$ be a right zero divisor. Then for any $\epsilon > 0$, there are a nonempty clopen subset $U \subset G^{(0)}$ and a unitary element $u \in C_r^*(G)$ satisfying $\|\chi_U u a\| < \epsilon$. The analogous statement also holds true for left zero divisors.*

Proof. Obviously it suffices to show the claim for right zero divisors. Let a right zero divisor $a \in C_r^*(G)$ and $\epsilon > 0$ be given. We may assume $\|a\| = 1$ and $\epsilon < 1$. Choose an element $x \in C_r^*(G)$ of norm one with $xa = 0$. By replacing x by x^*x , we may assume x is positive. Since the canonical conditional expectation E is faithful, we have $E(x) \neq 0$.

Let η be a positive number less than $\rho := \frac{1}{2}\|E(x)\|$. Choose a positive element $x_0 \in C_c(G)$ satisfying $\|x - x_0\| < \eta$ and $\|x_0\| = 1$. Note that $\|E(x_0)\| > \rho$. Choose compact open G -sets V_1, \dots, V_n whose union covers $\text{supp}(x_0)$. Take $s \in G^{(0)}$ satisfying $E(x_0)(s) > \rho$. Then, by Lemma 3.10, one can choose a clopen neighborhood $U \subset G^{(0)}$ of s satisfying $W := \bigcup_{i=1}^n \tau_{V_i}(U) \neq G^{(0)}$. Note that W is a clopen set. Moreover, we have $E(x_0 \chi_U x_0)(s) > \rho^2$ and $x_0 \chi_U x_0 \leq \chi_W$. Define

$$y := \|x_0 \chi_U x_0\|^{-1} x_0 \chi_U x_0 \in C_c(G).$$

We then have $\|E(y)\| > \rho^2$ and $\|ya\| < \rho^{-2}\eta$. Therefore, by choosing $\eta = \rho^4\epsilon$, we obtain a positive element $y \in C_c(G)$ of norm one which satisfies the following conditions.

- (1) $\|E(y)\| > \rho^2$.
- (2) $\|ya\| < \rho^2\epsilon$.
- (3) There is a proper clopen subset $W \subset G^{(0)}$ satisfying $y \leq \chi_W$.

Note that $E(y^2) \geq E(y)^2$ hence $\|E(y^2)\| > \rho^4$. Then, since $y \in C_c(G)$ and G is essentially principal (by Lemma 3.7), one can find a nonnegative function $f \in C(G^{(0)})$ satisfying the following conditions (cf. [2]).

- (1) $\|f\| \leq \rho^{-2}$.

- (2) $fy^2f \in C(G^{(0)})$.
- (3) $\|fy^2f\| = 1$.
- (4) The set $U := \text{int}(\{s \in G^{(0)} : (fy^2f)(s) = 1\})$ is nonempty.

Indeed, since G is essentially principal, one can choose $s \in G^{(0)}$ satisfying $y^2(s) > \rho^4$ and

$$r^{-1}(\{s\}) \cap s^{-1}(\{s\}) \cap \text{supp}(y^2) = \{s\}.$$

By the last equality, one can find an open neighborhood $V \subset G^{(0)}$ of s satisfying

$$r^{-1}(V) \cap s^{-1}(V) \cap \text{supp}(y^2) \subset G^{(0)}.$$

From this relation, for any $g \in C_0(V)$, we have $gy^2g \in C(G^{(0)})$. Take a nonnegative function $g \in C_0(V)$ satisfying

$$(y^2(s))^{-1/2} < g(s) \leq \|g\| \leq \rho^{-2}.$$

We then have $gy^2g \in C(G^{(0)})$ and $(gy^2g)(s) > 1$. Set $D := \{t \in G^{(0)} : (gy^2g)(t) \geq 1\}$. Define $h := ((gy^2g)|_D)^{-1/2} \in C(D)$. Take a continuous extension $k \in C(G^{(0)})$ of h satisfying $k \geq 0$ and $\|k\| \leq 1$. Then the function $f := gk$ satisfies the desired conditions.

Take a nonempty open subset U_a of U satisfying $\text{cl}(U_a) \subset U$. We also fix an element $s \in G^{(0)} \setminus W$. Put $S := \langle s \rangle$ (see Section 2.4 for the notation). By Lemma 3.8, one can find compact open G -sets W_1, \dots, W_m satisfying $S \subset \bigcup_{i=1}^m \tau_{W_i}(U_a)$, $S \subset r(W_i)$ for all i , and $\tau_{W_i}^{-1}(S) \cap \tau_{W_j}^{-1}(S) = \emptyset$ for $i \neq j$. Then by Lemma 2.5, one can take pairwise disjoint clopen subsets Z_1, \dots, Z_m of $G^{(0)}$ satisfying $\tau_{W_i}^{-1}(S) \subset Z_i$ for each i . For each i , set $U_i := U_a \cap Z_i$. Then the closures of U_1, \dots, U_m in $G^{(0)}$ are pairwise disjoint. Moreover, for each i , we have $S \cap \tau_{W_i}(U_a) = S \cap \tau_{W_i}(U_i)$. This yields $S \subset \bigcup_{i=1}^m \tau_{W_i}(U_i)$. Since the union $\bigcup_{i=1}^m \tau_{W_i}(U_i)$ is open, thanks to Lemma 2.4, one can find a clopen neighborhood $Z \subset G^{(0)} \setminus W$ of s contained in $\bigcup_{i=1}^m \tau_{W_i}(U_i)$.

Choose nonnegative functions $\rho_1, \dots, \rho_m \in C(G^{(0)})$ such that $\rho_i \rho_j = 0$ for $i \neq j$, that $\rho_i \equiv 1$ on U_i for each i , and that $\text{supp}(\rho_i) \subset \text{cl}(U)$ for each i . Set

$$v := \sum_{i=1}^m \chi_{W_i} \rho_i \in C_c(G).$$

We then have

$$\begin{aligned} vfy^2fv^* &= \sum_{1 \leq i, j \leq m} \chi_{W_i} \rho_i fy^2f \rho_j \chi_{W_j}^* \\ &= \sum_{i=1}^m \chi_{W_i} \rho_i^2 \chi_{W_i}^* \\ &= \sum_{i=1}^m \rho_i^2 \circ \tau_{W_i}^{-1} \\ &\geq \chi_Z. \end{aligned}$$

Moreover, this computation shows $vfy^2fv^* \in C(G^{(0)})$. Hence there is a function $g \in C(G^{(0)})$ of norm at most one satisfying $gvfy^2fv^*g^* = \chi_Z$. Take such a function g and put $w := gv$. We claim that w is of norm one. Indeed, since $\rho_i = \rho_i fy^2f$ for all i , we have $v = vfy^2f$. This shows $w = wfy^2f$. Hence we have

$$ww^* = wfy^2fw^* = \chi_Z.$$

In particular this yields $\|w\| = 1$.

Set $t := yfw^*$. Then t is a partial isometry element of $C_r^*(G)$ satisfying $t^*t = \chi_Z$ and $tt^* \leq \chi_W$. Since Z and W are disjoint, the projections t^*t and tt^* are orthogonal. Hence the element

$$u := t + t^* + (1 - t^*t - tt^*)$$

is a unitary element of $C_r^*(G)$. Moreover, by definition, we have

$$u^*\chi_Z u = tt^* = yfw^*wfy.$$

This shows

$$\begin{aligned} \|\chi_Z u a\| &= \|yfw^*wfy a\| \\ &\leq \|yf\| \|w^*w\| \|f\| \|ya\| \\ &< \epsilon. \end{aligned}$$

Hence Z and u satisfy the required condition. \square

The next lemma can be seen as a very special case of Green's imprimitivity theorem [12]. Since we need an explicit isomorphism, we include a proof.

Lemma 4.2. *Let K be a compact groupoid with a fundamental domain. Let $K^{(0)} = \bigsqcup_{i=1}^n \bigsqcup_{j=1}^{N_i} F_j^{(i)}$ be a partition of $K^{(0)}$ as in Definition 3.2. Then there is a $*$ -isomorphism*

$$\Phi: C_r^*(K) \rightarrow \bigoplus_{i=1}^n M_{N_i}(C(F_1^{(i)})).$$

Proof. For each $i \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, N_i\}$, let $V_{j,k}^{(i)}$ be the unique compact open K -set satisfying $r(V_{j,k}^{(i)}) = F_j^{(i)}$ and $s(V_{j,k}^{(i)}) = F_k^{(i)}$. We then have

$$K = \bigsqcup_{i=1}^n \bigsqcup_{1 \leq j, k \leq N_i} V_{j,k}^{(i)}.$$

This shows that the set

$$\left\{ f\chi_{V_{j,k}^{(i)}} : 1 \leq i \leq n, 1 \leq j, k \leq N_i, f \in C(F_j^{(i)}) \right\}$$

linearly spans $C(K) = C_r^*(K)$. For each $i \in \{1, \dots, n\}$, let $(e_{j,k}^{(i)})_{1 \leq j, k \leq N_i}$ denote the standard matrix units of $M_{N_i}(\mathbb{C}) \subset M_{N_i}(C(F_1^{(i)})) \subset \bigoplus_{i=1}^n M_{N_i}(C(F_1^{(i)}))$. Here the first inclusion is the canonical unital one. Define a linear map

$$\Phi: C_r^*(K) \rightarrow \bigoplus_{i=1}^n M_{N_i}(C(F_1^{(i)}))$$

to be

$$\Phi(f\chi_{V_{j,k}^{(i)}}) := (f \circ \tau_{V_{j,1}^{(i)}}) \cdot e_{j,k}^{(i)}$$

for i, j, k , and $f \in C(F_j^{(i)})$. Then direct computations show that Φ is a $*$ -isomorphism. \square

Proof of the Main Theorem. By Lemma 3.6, the reduced groupoid C^* -algebra $C_r^*(G)$ admits a faithful tracial state. Hence, thanks to Proposition 3.2 of [24], it suffices to show that any two-sided zero divisor of $C_r^*(G)$ is contained in $\overline{GL}(C_r^*(G))$.

Let a be a two-sided zero divisor of $C_r^*(G)$. We may assume a is of norm one. Let $\epsilon > 0$ be given. Then by Lemma 4.1, there are unitary elements u, v of $C_r^*(G)$ and nonempty clopen subsets $U, V \subset G^{(0)}$ satisfying $\|\chi_U u a v\| < \epsilon$ and $\|u a v \chi_V\| < \epsilon$. By Lemmas 3.10

and 3.11, one can choose a nonempty clopen subset U_0 of U and $\varphi \in [[G]]$ satisfying $\varphi(U_0) \subset V$. Therefore, by replacing U by U_0 and replacing v by vu_φ^* , we may assume $U = V$. Choose $b_0 \in C_c(G)$ satisfying $\|b_0\| \leq 1$ and $\|uav - b_0\| < \epsilon$. Set

$$b := b_0 - \chi_U b_0 - b_0 \chi_U + \chi_U b_0 \chi_U.$$

Since χ_U is a projection, we have $\chi_U b = b \chi_U = 0$.

By Lemma 3.10, one can find a nonempty clopen subset U_0 of U satisfying $\mu(U_0) < \frac{1}{2}\mu(U)$ for all $\mu \in M(G)$. We fix such a clopen set U_0 . Define

$$\rho := \inf\{\mu(U_0) : \mu \in M(G)\}.$$

Note that ρ is a positive number since $M(G)$ is nonempty (by Lemma 3.10) and weak-* compact.

Take a compact open neighborhood B of $\text{supp}(b)$. We claim that there is an elementary subgroupoid K of G satisfying the inequalities

$$\mu(s(B \setminus K)) < \mu(U_0) < \frac{1}{2}\mu(U)$$

for all $\mu \in M(K)$. To lead to a contradiction, assume that such a K does not exist. Set

$$\Lambda := \{(C, \eta) : C \subset G \text{ compact}, \eta > 0\}.$$

We equip Λ with the partial order \leq defined by $(C, \eta) \leq (C', \eta')$ if they satisfy the relations $C \subset C'$ and $\eta \geq \eta'$. This makes Λ a directed set. For each $\lambda = (C, \eta) \in \Lambda$, take a (C, η) -invariant elementary subgroupoid K_λ of G . Since B is realized as the union of finitely many compact G -sets, by Lemma 3.5, for sufficiently large $\lambda \in \Lambda$, we have

$$\mu(s(B \setminus K_\lambda)) \leq \frac{1}{2}\rho$$

for all $\mu \in M(K_\lambda)$. Therefore, without loss of generality, we may assume that each K_λ satisfies the above inequality. By assumption, for each $\lambda \in \Lambda$, we must have a K_λ -invariant probability measure μ_λ satisfying either $\mu_\lambda(U_0) \leq \frac{1}{2}\rho$ or $\mu_\lambda(U_0) \geq \frac{1}{2}\mu_\lambda(U)$. By Lemma 3.6, any weak-* cluster point of the net $(\mu_\lambda)_{\lambda \in \Lambda}$ is G -invariant. Since both U_0 and U are clopen, the above condition yields that either $\mu(U_0) \leq \frac{1}{2}\rho$ or $\mu(U_0) \geq \frac{1}{2}\mu(U)$ holds for any weak-* cluster point μ of $(\mu_\lambda)_{\lambda \in \Lambda}$. In either case, this is a contradiction.

Take an elementary subgroupoid K of G satisfying the inequalities

$$\mu(s(B \setminus K)) < \mu(U_0) < \frac{1}{2}\mu(U)$$

for all $\mu \in M(K)$. Since $s(B \setminus K)$ is a clopen subset of $G^{(0)}$, by Lemma 3.11, one can find a clopen subset V_0 of $G^{(0)} \setminus U_0$ which is equivalent to U_0 in K and satisfies

$$s(B \setminus K) \subset U_0 \sqcup V_0.$$

By the above inequalities, we further obtain

$$\mu(U \setminus (U_0 \sqcup V_0)) \geq \mu(U) - 2\mu(U_0) > 0$$

for all $\mu \in M(K)$. This inequality shows that the clopen set $U \setminus (U_0 \sqcup V_0)$ contains a fundamental domain of K .

Take an element $\varphi \in [[G]]$ satisfying $\varphi(U_0) = V_0$, $\varphi(V_0) = U_0$, and $\varphi(s) = s$ for all $s \in G^{(0)} \setminus (U_0 \sqcup V_0)$. Set $p := \chi_{V_0}$, $q := \chi_{U_0}$, and $w := u_\varphi$. Set $W := G^{(0)} \setminus (U_0 \sqcup V_0)$ and $e := 1 - p - q$. Note that $we = ew = e$. By the relation $s(\text{supp}(b) \setminus K) \subset U_0 \sqcup V_0$, we have

$ebe \in C_r^*(K_W)$. We also have $pwb = wqb = 0$ and $wbq = 0$. Therefore the matrix form of the element wb with respect to the decomposition $1 = p + e + q$ is of the form

$$wb = \begin{pmatrix} 0 & 0 & 0 \\ * & d & 0 \\ * & * & 0 \end{pmatrix},$$

where $d := ewbe = ebe$.

We claim that $d \in \overline{GL}(C_r^*(G_W))$. This yields that wb , hence b , is contained in $\overline{GL}(C_r^*(G))$. Since

$$\|b - uav\| \leq \|b - b_0\| + \|b_0 - uav\| < 7\epsilon,$$

this completes the proof.

To prove the claim, take a fundamental domain Z of K contained in $U \setminus (U_0 \sqcup V_0)$. Since Z is contained in W , it is also a fundamental domain of K_W . Note that since $Z \subset U$, we have $\chi_Z b = b\chi_Z = 0$. Take a partition $W = \bigsqcup_{i=1}^n \bigsqcup_{k=1}^{N_i} Z_k^{(i)}$ of W as in Definition 3.2 for K_W with $\bigsqcup_{i=1}^n Z_1^{(i)} = Z$. Put $N := \max\{N_i : i = 1, \dots, n\}$. By Lemma 3.10, for each i , one can find a partition

$$Z_1^{(i)} = \bigsqcup_{j=1}^{M_i} \bigsqcup_{k=1}^{L_{i,j}} Z_k^{(i,j)}$$

of $Z_1^{(i)}$ by clopen subsets satisfying $Z_k^{(i,j)} \sim Z_l^{(i,j)}$ in G_W and $L_{i,j} \geq N$ for all i, j, k, l . For each (i, j) , take a compact open G_W -set $W_{i,j}$ satisfying the following conditions.

- (1) $r(W_{i,j}) = s(W_{i,j}) = \bigsqcup_{k=1}^{L_{i,j}} Z_k^{(i,j)}$.
- (2) $\tau_{W_{i,j}}(Z_k^{(i,j)}) = Z_{k+1}^{(i,j)}$ for all $k \pmod{L_{i,j}}$.
- (3) $W_{i,j}^{L_{i,j}} = \bigsqcup_{k=1}^{L_{i,j}} Z_k^{(i,j)}$.

Define L to be the subgroupoid of G_W generated by K_W and $\bigcup_{i,j} W_{i,j}$. Since Z is a fundamental domain of K_W , it is not hard to show that the L is compact and that the union $\bigsqcup_{i,j} Z_1^{(i,j)}$ is a fundamental domain of L . Take a $*$ -isomorphism

$$\Phi: C_r^*(L) \rightarrow \bigoplus_{i,j} M_{N_i L_{i,j}}(C(Z_1^{(i,j)}))$$

as in the proof of Lemma 4.2. The equalities $\chi_Z d = d\chi_Z = 0$ show that $\Phi(f)$ is unitary equivalent to an element of the form

$$\bigoplus_{i,j} \text{diag}(0_{L_{i,j}}, c_{i,j}),$$

where $c_{i,j} \in M_{L_{i,j}(N_i-1)}(C(Z_1^{(i,j)}))$ for each (i, j) . Moreover, since $d \in C_r^*(K_W)$, for each i, j, k , the d commutes with the characteristic function of $r(K_W Z_k^{(i,j)})$, hence with that of $r(K_W Z_k^{(i,j)}) \cap (W \setminus Z)$. Since Z is a fundamental domain of K_W , for each i, j, k , the intersection $r(K_W Z_k^{(i,j)}) \cap (W \setminus Z)$ decomposes into the disjoint union of $(N_i - 1)$ clopen subsets of $G^{(0)}$ each of which is equivalent to $Z_k^{(i,j)}$ in K_W (hence in L). This proves that $c_{i,j}$ is unitary equivalent to an element of the form $\text{diag}(c_{i,j,1}, \dots, c_{i,j,L_{i,j}})$, where $c_{i,j,k} \in M_{N_i-1}(C(Z_1^{(i,j)}))$ for all k . Since $L_{i,j} \geq N > N_i - 1$ for all i, j , Lemma 4.2 of [8] now proves $d \in \overline{GL}(C_r^*(L)) \subset \overline{GL}(C_r^*(G_W))$. \square

Remark 4.3. The Main Theorem and Theorem 2.3 show that the reduced groupoid C^* -algebras of almost finite groupoids have cancellation of projections. From this, with minor modifications, one can adapt the arguments in Sections 3 and 4 of [18] to minimal almost finite totally disconnected groupoids. Consequently, for such a groupoid G , the reduced groupoid C^* -algebra $C_r^*(G)$ has real rank zero and strict comparison.

5. APPLICATIONS OF MAIN THEOREM TO TOPOLOGICAL DYNAMICAL SYSTEMS

In this section, we give applications of the Main Theorem. We first prove Corollary I. The following key lemma is a minor modification of Lemma 6.3 in [15].

Lemma 5.1. *Let Γ be an abelian group. Let $\alpha: \Gamma \curvearrowright X$ be a free action of Γ on a totally disconnected compact space. Then the transformation groupoid $X \rtimes_\alpha \Gamma$ is almost finite.*

Proof. We first observe that when a groupoid G is realized as the increasing union of almost finite open subgroupoids, then G itself is almost finite. Hence we only need to show the claim in the case Γ is finitely generated. In this case, we have a free abelian subgroup $\Lambda \leq \Gamma$ and a finite subgroup $F \leq \Gamma$ satisfying $\Gamma = \Lambda \oplus F$. Note that Λ is finitely generated hence isomorphic to \mathbb{Z}^d for some nonnegative integer d . Since F is finite, the quotient space $Y := X/F$ is totally disconnected. Since F commutes with Λ , the action α induces the action $\beta: \Lambda \curvearrowright Y$. The freeness of α implies that of β .

Let $\pi: X \rtimes_\alpha \Lambda \rightarrow Y \rtimes_\beta \Lambda$ denote the canonical quotient homomorphism. By Lemma 6.3 of [15], the transformation groupoid $Y \rtimes_\beta \Lambda$ is almost finite. Let $S \subset \Lambda$ be a finite subset and let $\epsilon > 0$. Take an $(S \times Y, \epsilon)$ -invariant elementary subgroupoid $K \subset Y \rtimes_\beta \Lambda$. Set $\tilde{K} := \pi^{-1}(K)$. Since π is proper, \tilde{K} is an elementary subgroupoid of $X \rtimes_\alpha \Lambda$. By the definition of \tilde{K} , we have the equality

$$(\{g\} \times X) \cdot \tilde{K} \cdot (\{g^{-1}\} \times X) = \tilde{K}$$

for all $g \in F$. Let L be the subgroupoid of $X \rtimes_\alpha \Gamma$ generated by \tilde{K} and $F \times X$. Then L is an $((SF) \times X, \epsilon)$ -invariant elementary subgroupoid of $X \rtimes_\alpha \Gamma$. \square

Lemma 5.2. *Let G be a groupoid which admits a quotient homomorphism π from G onto an almost finite groupoid H with the following properties: π is proper, and the restriction map $\pi|_{Gs}: Gs \rightarrow H\pi(s)$ is bijective for each $s \in G^{(0)}$. Then G is almost finite.*

Proof. By the assumptions on π , for any elementary subgroupoid K of H , the preimage $\pi^{-1}(K)$ is an elementary subgroupoid of G . Furthermore, for any compact subset $C \subset H$ and $\epsilon > 0$, K is (C, ϵ) -invariant if and only if $\pi^{-1}(K)$ is $(\pi^{-1}(C), \epsilon)$ -invariant. \square

Proof of Corollary I. Let $\alpha: \Gamma \curvearrowright X$ be an action as in the statement. Then observe that, since Γ is abelian, the factor of α corresponding to the C^* -subalgebra of $C(X)$ generated by projections is minimal and free. The statement now follows from the Main Theorem and Lemmas 5.1 and 5.2. \square

We close this section by dealing with topological dynamical systems of amenable groups.

We first recall some terminologies related to partitions of underlying spaces of group actions. Let $\alpha: \Gamma \curvearrowright X$ be an action. A tower of α is a pair (S, B) of a nonempty finite subset $S \subset \Gamma$ and a nonempty subset $B \subset X$ such that the sets sB ; $s \in S$, are pairwise disjoint. The set S is called the shape of the tower (S, B) . A castle of α is a sequence $(S_1, B_1), \dots, (S_n, B_n)$ of towers such that the sets $S_1 B_1, \dots, S_n B_n$ are pairwise disjoint.

The sets S_1, \dots, S_n are called the shapes of the castle $((S_i, B_i))_{i=1}^n$. Let F be a finite subset of Γ and let $\epsilon > 0$. A finite subset S of Γ is said to be (F, ϵ) -invariant if it satisfies

$$\sharp(FS \setminus S) < \epsilon \sharp(S).$$

A castle $((S_i, B_i))_{i=1}^n$ is said to be (F, ϵ) -invariant if all the shapes S_1, \dots, S_n are (F, ϵ) -invariant. A castle $((S_i, B_i))_{i=1}^n$ is said to be clopen if each B_i is clopen. We say that a castle $((S_i, B_i))_{i=1}^n$ is full if the union $\bigsqcup_{i=1}^n S_i B_i$ covers X .

Lemma 5.3. *Let $\alpha: \Gamma \curvearrowright X$ be an action. Let $F \subset \Gamma$ be a finite subset and let $\epsilon > 0$. Then $X \rtimes_\alpha \Gamma$ admits an $(F \times X, \epsilon)$ -invariant elementary subgroupoid if and only if α admits an (F, ϵ) -invariant full clopen castle. Consequently, the transformation groupoid $X \rtimes_\alpha \Gamma$ is almost finite if and only if α admits full clopen castles with arbitrary invariance.*

Proof. Put $G := X \rtimes_\alpha \Gamma$. To prove the claim, we first give a correspondence between elementary subgroupoids of G and full clopen castles of α .

Let K be an elementary subgroupoid of G . Take a partition

$$K = \bigsqcup_{i=1}^n \bigsqcup_{1 \leq j, k \leq N_i} V_{j,k}^{(i)}$$

of K as in Definition 3.2. By discarding empty sets, we may assume that each $V_{j,k}^{(i)}$ is nonempty. Also, by refining the partition if necessary, we may assume that each $V_{j,k}^{(i)}$ is of the form $\{s\} \times U$; $s \in \Gamma$, $U \subset X$. Put $B_i := V_{1,1}^{(i)}$ for each i . Let $\pi: G \rightarrow \Gamma$ denote the projection onto the first coordinate. Set $S_i := \pi(s^{-1}(B_i) \cap K) \subset \Gamma$ for each i . Then it is clear from the definition that the sequence $((S_i, B_i))_{i=1}^n$ is a full clopen castle of α .

Conversely, if we have a full clopen castle $((S_i, B_i))_{i=1}^n$ of α , then the formula

$$K := \bigsqcup_{i=1}^n \bigsqcup_{s, t \in S_i} \{st^{-1}\} \times tB_i$$

defines an elementary subgroupoid of G .

We observe that, up to refinements of clopen castles, these correspondences are each other's inverse. Now one can show that a full clopen castle is (F, ϵ) -invariant if and only if the corresponding elementary subgroupoid is (F, ϵ) -invariant. This proves the statement. \square

Kerr recently announced the following theorem [13].

Theorem 5.4 ([13]). *For any countable amenable group, its generic minimal free actions on the Cantor set admit full clopen castles with arbitrary invariance.*

Corollary II now follows from the Main Theorem, Lemmas 5.2, 5.3, and Theorem 5.4.

Although the question on genericity no longer makes a sense, by adapting Theorem 4.3 of [6] to standard arguments of extensions, one can prove the following statement.

Theorem 5.5. *Let Γ be an amenable group. Then there is a totally disconnected compact space X with the following properties.*

- (1) X has a basis of cardinality at most $\sharp(\Gamma)$.
- (2) There is a minimal free action of Γ on X which admits full clopen castles with arbitrary invariance.

Consequently, we obtain the following result for the universal minimal actions [7] of amenable groups.

Corollary 5.6. *For any amenable group, the crossed product of the universal minimal action has stable rank one.*

APPENDIX A. PURE INFINITENESS OF CROSSED PRODUCTS FOR MINIMAL EXTENSIONS

In this appendix, we adapt arguments in the proof of Lemma 4.1 to the study of the pure infiniteness of the reduced groupoid C^* -algebras. As an application, we solve the question of the pure infiniteness of the reduced crossed product for minimal extensions.

Recall that a simple unital C^* -algebra A is said to be purely infinite [5] if any nonzero positive element $a \in A$ admits an element $x \in A$ satisfying $axa^* = 1$. The following theorem is the main result of this appendix.

Theorem A.1. *Let G be a minimal groupoid which admits a quotient homomorphism $\pi: G \rightarrow H$ satisfying the following conditions.*

- (1) *H is essentially principal.*
- (2) *π is proper.*
- (3) *For each $s \in G^{(0)}$, the restriction map $\pi|_{Gs}: Gs \rightarrow H\pi(s)$ is bijective.*
- (4) *The reduced groupoid C^* -algebra $C_r^*(H)$ is purely infinite.*

Then the reduced groupoid C^ -algebra $C_r^*(G)$ is purely infinite.*

Proof. By the second and third assumptions, the pull-back map $\pi^*: C_c(H) \rightarrow C_c(G)$ of π defines a $*$ -homomorphism. It is not hard to show that the map π^* extends to an embedding $C_r^*(H) \rightarrow C_r^*(G)$. Throughout the proof, we identify $C_r^*(H)$ with a C^* -subalgebra of $C_r^*(G)$ via this embedding.

Let $a \in C_r^*(G)$ be a positive element satisfying $\|E(a)\| = 1$. Choose a positive element $a_0 \in C_c(G)$ satisfying $\|a - a_0\| < 1$ and $\|E(a_0)\| > 1$. Since G is essentially principal (by the first to third assumptions), as in the proof of Lemma 4.1, one can find a nonnegative function $f \in C(G^{(0)})$ satisfying $\|f\| \leq 1$, $g := fa_0f \in C(G^{(0)})$, $\|g\| = 1$, and

$$U := \text{int}(\{s \in G^{(0)} : g(s) = 1\}) \neq \emptyset.$$

Fix a nonempty open subset U_0 of U satisfying $\text{cl}(U_0) \subset U$. Take compact H -sets $\bar{V}_1, \dots, \bar{V}_n, \bar{W}_1, \dots, \bar{W}_n$ satisfying the following conditions: $\bar{W}_i \subset \text{int}(\bar{V}_i)$ for all i and $G^{(0)} = \bigcup_{i=1}^n r(W_i U_0)$, where $W_i := \pi^{-1}(\bar{W}_i)$ for each i .

Note that by the second and third assumptions, each W_i is a compact G -set. Since H is essentially principal, one can find an element $s \in H^{(0)}$ satisfying

$$r^{-1}(\{s\}) \cap s^{-1}(\{s\}) \cap \bar{V}_i \bar{V}_j^{-1} \subset \{s\}$$

for all i, j . Put $S := \pi^{-1}(\{s\})$. Then the equalities $r(W_i) = \pi^{-1}(r(\bar{W}_i))$; $i \in \{1, \dots, n\}$, yield the relation $S \subset \bigcup_{i \in I} \tau_{W_i}(U_0)$, where

$$I := \{i \in \{1, \dots, n\} : s \in r(\bar{W}_i)\}.$$

Take a maximal subset J of I satisfying the following property: the elements $\tau_{\bar{W}_j}^{-1}(s)$; $j \in J$ are pairwise distinct. By the choice of s and the third assumption, for any $i, j \in \{1, \dots, n\}$,

$$r^{-1}(S) \cap s^{-1}(S) \cap V_i V_j^{-1} \subset S,$$

where $V_i := \pi^{-1}(\bar{V}_i)$ for each i . From the above relations, for any $i, j \in I$, the equality $\tau_{\bar{W}_i}^{-1}(s) = \tau_{\bar{W}_j}^{-1}(s)$ shows the equality

$$\tau_{W_i}(U_0) \cap S = \tau_{W_j}(U_0) \cap S.$$

This yields the relation $S \subset \bigcup_{j \in J} \tau_{W_j}(U_0)$.

Since the compact sets $\tau_{W_j}^{-1}(S); j \in J$, are pairwise disjoint, one can find nonnegative functions $\rho_j \in C(G^{(0)}); j \in J$ satisfying $\rho_j \rho_k = 0$ for $j \neq k$, $\rho_j \equiv 1$ on a neighborhood of $\tau_{W_j}^{-1}(S) \cap U_0$ for each $j \in J$, and $\text{supp}(\rho_j) \subset \text{cl}(U)$ for all $j \in J$. For each $j \in J$, take a function $\varphi_j \in C_c(G)$ satisfying $\varphi_j \equiv 1$ on a neighborhood of W_j and $\text{supp}(\varphi_j) \subset V_j$. Set $v := \sum_{j \in J} \varphi_j \rho_j$. Then, since $\rho_j = \rho_j g$ for all $j \in J$, we have $vg = v$. This implies

$$vgv^* = \sum_{j \in J} \varphi_j \rho_j^2 \varphi_j^*,$$

This equality shows that $vgv^* \in C(G^{(0)})$ and that the inequality $(vgv^*)(t) \geq 1$ holds on a neighborhood W of S . Then, by the compactness of $G^{(0)}$, one can choose a neighborhood \bar{Z} of s satisfying $Z := \pi^{-1}(\bar{Z}) \subset W$. Take a function $h \in C(G^{(0)})$ of norm at most one satisfying $(hvgv^*h^*)(t) = 1$ for all $t \in Z$. Take functions $k_1, k_2 \in C(H^{(0)})$ of norm one satisfying $k_1 k_2 = k_2$ and $\text{supp}(k_1) \subset \bar{Z}$. We then have

$$k_1 hvgv^* h^* k_1^* = |k_1|^2 \in C(H^{(0)}).$$

Put $w := k_1 h v$. Then the above equality yields $ww^* = wgw^* = |k_1|^2$.

Since $C_r^*(H)$ is purely infinite, one can find an element $y \in C_r^*(H)$ satisfying $y|k_2|^2 y^* = 1$. Set $x := yk_2 w f$. Then, the equalities $k_1 k_2 = k_2$ and $\|f\| = 1$ yield

$$xx^* \leq yk_2 ww^* k_2^* y^* = y|k_2|^2 y^* = 1.$$

This shows $\|x\| \leq 1$. Moreover, by the equality $k_1 k_2 = k_2$, we have

$$xa_0 x^* = yk_2 wgw^* k_2^* y^* = y|k_2|^2 y^* = 1.$$

This yields $\|xax^* - 1\| < 1$, from which we conclude the invertibility of xax^* . \square

Corollary A.2. *For minimal essentially free actions of a group on a compact space, pure infiniteness of the reduced crossed product passes to minimal extensions.*

Corollary A.2 has the following two immediate applications. As the first application, we give the following stronger version of Theorem 3.11 in [26]. This is a new phenomenon in the study of amenable actions ([1]).

Corollary A.3. *Any countable nonamenable group admits a minimal free action on the Cantor set whose all minimal extensions have the purely infinite reduced crossed product. Moreover, for exact groups, such actions can be chosen to be amenable.*

Since a statement similar to Theorem 2.3 holds true for purely infinite C^* -algebras (see Theorems 1.4 and 1.9 of [5]), Corollary A.3 can also be seen as an analog of Corollary II for nonamenable groups.

The second application is the removal of a technical restriction of Theorem 3.12 in [27].

Corollary A.4. *Let X denote the Cantor set. Let Y be a compact metrizable space which admits a minimal action of a path connected group. Put $Z := X \times Y$. Then any countable nonamenable exact group admits an amenable minimal free action on Z whose reduced crossed product is a Kirchberg algebra.*

Proof. The statement immediately follows from Theorem 2.1 of [27] and Corollary A.3. \square

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GRADUATE SCHOOL OF SCIENCE, CHIBA UNIVERSITY, INAGE-KU, CHIBA 263-8522, JAPAN
E-mail address: suzukiyu@ms.u-tokyo.ac.jp